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Antenna Study for Z-mode Space Radio Program

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Summary

The antenna system for the Z-mode study must not only act as a receptor for cosmic signals but must also be capable of measuring the local properties of the ionospheric medium. The reception properties of the antenna have been investigated and the effective area formulas are presented. The measurement of the plasma parameters (i.e., electron density N and collision frequency ν) is complicated by the proximity of the operating frequency (for the Z-mode) to the cyclotron frequency. By operating sufficiently above the cyclotron frequency, the anisotropy in the medium may be neglected. The theoretical relation between the base impedance of the antenna probe and the properties of the medium determines N and ν. The collision frequency, however, is too small to be measured when the operating frequency is elevated above the cyclotron frequency. A theory has been developed which allows the antenna to operate near the cyclotron frequency. The special case of the dipole alligned along the magnetic field has been computed. A preliminary antenna study would aid in determining the validity of the theory.

An alternative method is available for measuring N and ν near the cyclotron frequency. In this method only the receiving properties of the antenna are needed. Since alternative methods are available for determining the actual plasma parameters the new theory may be easily checked.

ANTENNA STUDY FOR Z-MODE SPACE RADIO PROGRAM

1. Introduction

It has been suggested by Ellis [1] that an antenna physically located in the Z-mode propagation region of the terrestrial ionosphere would be subject to greatly increased angular resolution. Theoretical calculations show that the expected beamwidth would be the order of a fraction of a degree. This order of resolution is due to the special propagation properties of waves in an ionosphere containing a static magnetic field. That is, only those incoming waves which form a small cone about the direction of the magnetic field are allowed to illuminate the antenna. Simple ray tracing has already been applied to the problem of finding the cone angle of the incoming waves. However, more sophisticated techniques are needed to determine the actual antenna parameters.

One method of determining the electron density N, and collision frequency ν of a plasma is to measure the driving point impedance of the antenna [2]. This method has been successful for the case of a linear antenna immersed in an isotropic plasma (i.e., no magnetic field). However, for propagation of waves under Z-mode conditions the operational and cyclotron frequencies are necessarily of the same order. The antenna problem consists of relating antenna measurements to the parameters N and ν with the dipole immersed in

the anisotropic medium. For the case of the antenna immersed in an isotropic medium, the driving point impedance has been related to N and ν [3] . Unfortunately, the problem of finding the driving point impedance of an antenna in an anisotropic medium has not yet been solved. The complication is due to the difficulty in writing a single equation which involves the antenna current.

A further complication is found in the measurement of the collision frequency. In the region of interest, the collision frequency is very small compared to the plasma, cyclotron, and operating frequencies. Three possible measurement procedures will be presented in this paper.

In the first method, the measurement frequency is about ten times the cyclotron frequency. The anisotropy of the medium is then neglected in the theoretical calculation of the electron density and collision frequency. No difficulty arises in determining N from driving point impedance measurements on the antenna. However the measurement of ν is marginal, due to its small value in the ionospheric region of interest.

The second method utilizes measurements made at the Z-mode operating frequency. Here the anisotropy of the medium must be considered. An approximate method of determining N and ν is given. This method requires a knowledge only of the receiving properties of the linear antenna.

In the simplified model of the receiving system, it is assumed that all the anisotropic properties of the medium are contained in the wave impinging on the antenna. This physical model follows from the Thévenin equivalent circuit for a receiving antenna. The equivalent impedance of a receiving antenna is the impedance seen at the antenna terminals with all sources short circuited (e.g., no electric field from a distant source). The equivalent driving voltage is the open circuit voltage at the antenna terminals.

Along with the measurement of the plasma parameters N and ν, we must also determine the effective area of the antenna. The effective area is computed from the relation between the power density $S(wm^{-2})$ in the medium and the received power as measured at the terminals of a load impedance. The effective area is easily computed from the equivalent circuit of the receiving antenna in an anisotropic medium.

The most direct approach is to measure N and ν near the plasma frequency. The medium is sufficiently lossy near the plasma frequency for a dependable measurement of ν. In this case, since the plasma frequency is near the cyclotron frequency, the anisotropy of the medium must be considered. A new theory with some particular solutions is presented.

2. Determining the plasma parameters with anisotropy neglected

The driving point impedance ($Z_o = \frac{1}{Y_o}$) of an electrically short cyclindrical antenna immersed in a plasma medium has been theoretically calculated by King [4]. The electron density and collision frequency are related to the constitutive parameters of the medium, i.e., the dielectric constant ϵ and the conductivity σ. The theory relates the driving point impedance to the constitutive parameters of the medium. From the values of ϵ and σ the values of N and ν are easily determined. The representative values of N and ν for the Z-mode propagation study are:

$$
\begin{array}{ll}
 & N = 10^{11} \text{ electrons } m^{-3} \\
F_2 \text{ region of} & \nu = 5 \times 10^{2} \text{ cps} \\
\text{ionosphere} & \omega_b = \text{cyclotron frequency} = 8.6\text{x}10^{6} \text{ rad/sec} \\
 & \omega_p = \text{plasma freq} = 17.6\text{x}10^{6} \text{ rad/sec.}
\end{array}
\qquad (2\text{-}1)
$$

The proposed unfurlable cyclindrical antenna has the following dimensions:

$$
\begin{array}{l}
h = \text{half-length} = 4.68 \text{ ft} = 1.43 \text{ M} \\
a = \text{radius} = \frac{1}{4} \text{ inch} = (1/48) \text{ ft.} \\
\Omega = 2 \ln(2h/a) = 12.2 \quad .
\end{array}
\qquad (2\text{-}2)
$$

The above antenna may be considered as a representative short antenna for space probing operations. The operating frequency f_o is 10.0 mc/s. The electrical length in vacuo is $\beta_o h = \frac{2\pi}{\lambda_o} \cdot h = 0.30$, and the driving point impedance of the antenna in vacuo is given by King [5], as:

$$Y(\beta_o) = \frac{2\pi\beta_o^4 h^4}{3(\Omega-3)\zeta_o\Psi_{d1}} + j\,\frac{2\pi\beta_o h}{\zeta_o\Psi_{d1}}\left(1 + \frac{\beta_o^2 h^2 F}{3}\right). \quad (2\text{-}3)$$

Where

$$\begin{aligned} \beta_o h &= 0.30 < 1 \\ \Omega &= 2\,\ln(2h/a) = 12.2 \\ \Psi_{d1} &= 2\,\ln(h/a) - 2 = 8.8 \\ F &= 1 + (1.08/(r-3)) = 1.117, \end{aligned} \quad (2\text{-}4)$$

and with eqn.(2-4) in eqn.(2-3) the driving point admittance is:

$$Y(\beta_o) = .556 \times 10^{-6} + j\,0.590 \times 10^{-3} \text{ mho}$$

or

$$Z(\beta_o) = 1/Y(\beta_o) = 1.60 - j\,1690 \text{ ohms} \quad (2\text{-}5)$$

Note that the driving point reactance $Z(\beta_o)$ is mainly capacitive, a characteristic of short antennas (i.e., $\beta_o h < 1$). With the measurement frequency sufficiently elevated above the cyclotron frequency, the magnetic field may be neglected. The antenna may then be considered as immersed in a homogeneous, isotropic conducting dielectric (i.e., a plasma) characterized by an equivalent dielectric instant (ϵ) and conductivity (σ) given by the classical relations:

$$\epsilon = \epsilon_o\epsilon_r\left[1 - \frac{Ne^2}{\epsilon_o m(\nu^2+\omega^2)}\right] \text{farads per meter} \quad (2\text{-}6)$$

$$\sigma = \frac{Ne^2\nu}{m(\nu^2+\omega^2)} \text{ mho/m} \quad (2\text{-}7)$$

where

$\epsilon_o = 8.85 \times 10^{-12}$ farads per meter = dielectric constant in vacuo

ω = operating frequency $(2\pi f_o)$

e = charge on electron = 1.602×10^{-19} coulombs

m = mass of electron = 9.108×10^{-31} kg.

The values of ϵ and σ which correspond to eqn.(2-1) and $\omega = 20\pi \times 10^6$/sec $(\sim f_o = 10$ mc) are:

$$\begin{aligned} \epsilon &= 0.919\,\epsilon_o \\ \sigma &= 3.58 \times 10^{-10} \text{ mho/m} . \end{aligned} \tag{2-8}$$

The values in eqn.(2-8) correspond to a loss tangent $(= \frac{\sigma}{\omega \epsilon})$ of 7.01×10^{-7}. This low value for the loss tangent indicates that the plasma is only slightly conducting. The driving point admittance of the antenna immersed in the plasma is determined from formulas in King [6]. A comparison of the values in vacuo and in plasma indicates whether a measurable difference exists for determining both N and ν. At the outset it should be noted that the major problem is in measuring ν and not in measuring N. From the viewpoint of measuring ν, the normal arrangement would include a measuring system which operates near the collision frequency. However, since the collision frequency is far below the cyclotron frequency, the magnetic field terms are a major contribution to the condutivity. Furthermore the theorist must still contend with the anisotropy of the medium.

The driving point admittance of a short antenna ($\beta h < 1$) immersed in a plasma is given by: [N.B. the complete formula for $(\alpha/\beta) < 1$ is given in Ref. 4].

$$Y(k) = \frac{2\pi}{\zeta_e \Psi_{d1}} \left\{ \frac{2\alpha}{\beta} \left[\beta h + \frac{2}{3} \beta^3 h^3 F \right] + \frac{\beta^4 h^4}{3(\Omega - 3)} \right. \qquad (2\text{-}9)$$

$$\left. + j \left[\beta h + \frac{1}{3} \beta^3 h^3 F - \frac{5}{3} \frac{\alpha}{\beta} \frac{\beta^4 h^4}{\Omega - 3} \right] \right.$$

where $\left(2 \frac{\alpha}{\beta}\right)^2 << 1$

where

k = propagation constant = $\alpha - j\beta$

α = attenuation constant

β = phase constant

$\zeta_e = \zeta_o / \sqrt{\epsilon_r}$, $\beta = \beta_o \sqrt{\epsilon_r}$

also for $\left(\frac{2\alpha}{\beta}\right)^2 << 1$

$$k = \beta\left(1 - j\frac{\alpha}{\beta}\right) \approx \omega\sqrt{\mu\,\epsilon}\left(1 - j\frac{\sigma}{2\omega\epsilon}\right). \qquad (2\text{-}10)$$

The driving point admittance for the short antenna in plasma is given by eqn.(2-9) with eqns.(2-8) and (2-10), thus:

$$Y(k) = G(k) + j\,B(k); \qquad (2\text{-}11)$$

where

$$G(k) = 3.82 \times 10^{-10} + 4.49 \times 10^{-7} \text{ mho} \qquad (2\text{-}12)$$

$$B(k) = 5.65 \times 10^{-4} \text{ mho} \qquad (2\text{-}13)$$

The measurement of ν depends on an appreciable change in

conductance G with the dipole in vacuo and in the plasma. The value of G due to the change in dielectric constant alone is:

$$G(h) = \epsilon^{5/2}\, G\,(\beta_o) = 4.49 \times 10^{-7} \text{ mho} \qquad (2\text{-}14)$$

A comparison of eqns. (2-12) and (2-14) shows that the contribution due to the collision frequency (3.82×10^{-10} mho) is insufficient for measurement. The situation may be improved by operating closer to the cyclotron frequency. Actually the above calculations assume an operating frequency higher than may be necessary (i.e., $(\omega/\omega_b') = 7.25$) in order to neglect the anisotropy. The criterion for error introduced in neglecting the anisotropy of the medium is the ratio (η) of the off-diagonal to the diagonal complex dielectric coefficient [7], thus:

$$\eta = \frac{\left(\frac{\omega_b}{\omega}\right)\left(\frac{\omega_p}{\omega_b}\right)^2}{\left(\frac{\omega}{\omega_b}\right)^2 - 1 - \left(\frac{\omega_p}{\omega_b}\right)^2} \qquad (2\text{-}15)$$

where $\nu \ll \omega, \omega_b, \omega_p$.

For example, at an operating frequency of 10 mc/s ($\omega = 20\pi \cdot 10^6$ rad. per sec), $\omega = 8.6 \times 10^6$ rad. per sec., and $\omega_p = 17.6 \times 10^6$ rad per sec (see eqn. 2-1) then

$$\eta = 1.21 \times 10^{-2}\,. \qquad (2\text{-}16)$$

From the low value of η in eqn. (2-16) it is expected that the error in neglecting the anisotropy will be the order of one per cent.

In order to get a sufficient loss in the medium for a dependable measurement of ν it is necessary to operate close to the plasma frequency. For example, consider the value of (α/β) given by eqs. (2-10), (2-6) and (2-7), or

$$\left(\frac{\alpha}{\beta}\right) = \frac{\omega_p^{\,2}\nu}{2\omega(\nu^2+\omega^2-\omega_p^{\,2})} \qquad (2\text{-}17)$$

With $\beta h = 0.3$, the value of (α/β) must be approximately 10^{-4} in order to give a measurable change in conductance (see eq. 2-4). With the operating frequency reduced to 4.0 mc the corresponding value of (α/β) from eq. (2-17) is 9.60×10^{-6}. The admittance of the antenna in the plasma is given by eqn. (2-4) with $(\alpha/\beta) = 9.6 \times 10^{-6}$, $\epsilon_r = 0.544$ and $\beta h = \sqrt{\epsilon_r}\,\beta_o h = 0.221$, thus:

$$G(k) = (8.06 \times 10^{-9} + 2.17 \times 10^{-7}) = 2.25 \times 10^{-7}$$

$$Y(k) = +4.38 \times 10^{-4} \text{ mho}$$

or

$$Z(h) = 1.17 - j\,1002 \text{ ohms.}$$

The value of conductance due to the change in ϵ_r alone is 2.17×10^{-7}, and the contribution from the collision frequency term is 8.06×10^{-9}. This change in conductance is perhaps just sufficient for detection. However the error in the measurement due to proximity to the cyclotron

frequency is rather large. As a measure of the error, the anisotropy ratio η in eq. (2-15) becomes:

$$\eta = 0.45.$$

3. Approximate solution for the driving point impedance of an electric dipole immersed in an anisotropic plasma

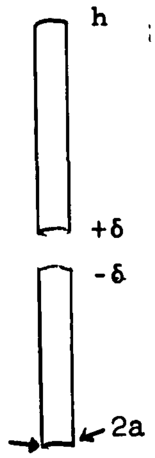

Fig. 3-1. Cylindrical antenna

The analysis in this paper is restricted to antennas where the length is such that

$$\beta_o h \ll 1 \tag{3-1}$$

where

$$\beta_o = 2\pi/\lambda_o. \tag{3-2}$$

The relevant boundary conditions require that the tangential components of the electric field are continuous at r = a or

$$\hat{n} \times \vec{E} = \hat{n}_1 \times \vec{E}_1 + \hat{n}_2 \times \vec{E}_2 = 0. \tag{3-3}$$

where $\hat{n}_1$ refers to the conductor and $\hat{n}_2$ to the medium. Equation (3-3) is equivalent to:

$$(E_{Z1})_{r=a} = (E_{Z2})_{r=a} \; . \tag{3-4}$$

Furthermore the current is considered to be entirely in the Z-direction and

$$I_Z(Z) = 0, \; Z = \pm h \; . \tag{3-5}$$

The antenna is driven by a slice generator of potential difference V_o where:

$$V_o = \lim_{\delta \to 0} \left[\phi(\delta) - \phi\,(-8) \right] = 2\,\phi\,(0). \tag{3-6}$$

In the medium the electric field is given by:

$$\vec{E}_2 = - \vec{\nabla}\,\phi_2 \; - j\omega \vec{A}_2 \tag{3-7}$$

The gauge condition may be used to relate the vector and scalar potential, thus for a tensor dielectric constant:

$$\vec{\nabla}\,(\vec{\nabla} \cdot \vec{A}_2) + j\omega \epsilon_o \overleftrightarrow{\epsilon}\,\vec{\nabla}\,\phi_2 = 0. \tag{3-8}$$

With the value of $\vec{\nabla}\,\phi_2$ in eq. (3-8) substituted in eq. (3-7) the electric field is given by:

$$E_Z = \frac{-j\overleftrightarrow{\epsilon}^{\,-1}}{\omega\,\epsilon_o}\,(\vec{\nabla}\,(\vec{\nabla} \cdot \vec{A}) + \omega^2 \epsilon_o \overleftrightarrow{\epsilon} A). \tag{3-9}$$

As $I = I_Z$ and $\vec{A} = A_Z$, then at $r = a$:

$$(E_{Z2})_{r=a} = \frac{-j\overleftrightarrow{\mathcal{E}}^{-1}}{\omega \epsilon_o} \left[\frac{\partial^2 A_Z}{\partial Z^2} + \beta_o^{\,2} \overleftrightarrow{\mathcal{E}} A_Z \right]. \tag{3-10}$$

The electric field just inside the conductor $(E_{Z1})_{r=a}$ is given in terms of internal impedance per unit length Z^1 and the total current I_Z, thus:

$$(E_{Z1})_{r=a} = Z^1 I_Z \tag{3-11}$$

Substituting eqs. (3-10) and (3-11) in eq. (3-4), it follows:

$$\frac{\partial^2 A_Z}{\partial Z^2} + \beta_o^{\,2} \overleftrightarrow{\mathcal{E}} A_Z = j \frac{\beta_o^{\,2}}{\omega} \overleftrightarrow{\mathcal{E}} Z^1 I_Z . \tag{3-12}$$

It should be noted in passing that the gauge condition in eq. (3-8) must be compatible with eq. (3-12). At the outset it is known that for good conductors $Z^1 \simeq 0$, and the particular solution of the differential equation can be neglected. We must therefore look for a solution of the homogeneous equation:

$$\frac{\partial^2 A_Z}{\partial Z^2} + \beta_o^{\,2} \overleftrightarrow{\mathcal{E}} A_Z = 0 . \tag{3-12}$$

The vector potential is expressed by:

$$\left\{\overleftrightarrow{A}\right\} = 1/4\pi\nu_o \int_V \overleftrightarrow{G} (\vec{r}\ \vec{r}^{\,1}) \left\{I\right\} d\,v . \tag{3-13}$$

where $\overleftrightarrow{G}$ = tensor Green's function for an anisotropic medium. For our case:

$$\left\{I\right\} = \left\{\begin{matrix} 0 \\ 0 \\ I_Z \end{matrix}\right\}. \tag{3-14}$$

The tensor Green's function is obtained by the following inverse transform [8]:

$$\overleftrightarrow{G} = \frac{1}{(2\pi)^3} \int_{\vec{K}} \overleftrightarrow{\lambda}^{-1} e^{-i\vec{k} \cdot (\vec{r}-\vec{r}^1)} dk \tag{3-15}$$

where

$$\overleftrightarrow{\lambda} = k_o^2 \overleftrightarrow{\epsilon} - k^2 \overleftrightarrow{I}$$

$$\overleftrightarrow{I} = \text{identity tensor.} \tag{3-16}$$

As a specific example consider the case where the magnetic field is parallel to the antenna. The tensor dielectric constant is given by:

$$\left[\overleftrightarrow{\epsilon}\right] = \begin{bmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ j\epsilon_{12} & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} . \tag{3-17}$$

The $\overleftrightarrow{G}$ tensor has already been computed for this case by Cook [9]. From eq. (3-13) it follows that:

$$A_Z = 1/4M\nu_o \int_{-h}^{h} G_{33}(Z_1Z^1)\, I_Z(Z^1)dZ^1 \tag{3-18}$$

where

$$G_{33}(Z_1Z^1) = e^{-ik_3Z/R}$$

$$k_3 = \beta_o \epsilon_{33}^{\frac{1}{2}}$$

$$R = \sqrt{(Z-Z^1)^2 + a^2} . \tag{3-19}$$

The form of the vector potential eq. (3-18) and the homogeneous Helmholtz equation (3-12) is identical to the isotropic case with

$$\epsilon = \epsilon_{33} . \tag{3-20}$$

4. Theory for the small linear receiving antenna in a plane wave field

For this analysis the length of the antenna is considered to be small compared with a wavelength, that is,

$$\beta_0 h < 1 \tag{4-1}$$

where

$\beta = 2\pi/\lambda_0$, h = half length of antenna.

This condition not only simplifies the mathematical formulation but insures that the properties of the medium do not change over the length of the receiving antenna. For the frequencies of interest the length of the antenna compared with a wavelength is indeed small.

The receiving antenna in a plane wave field is shown in Fig. 4-1.

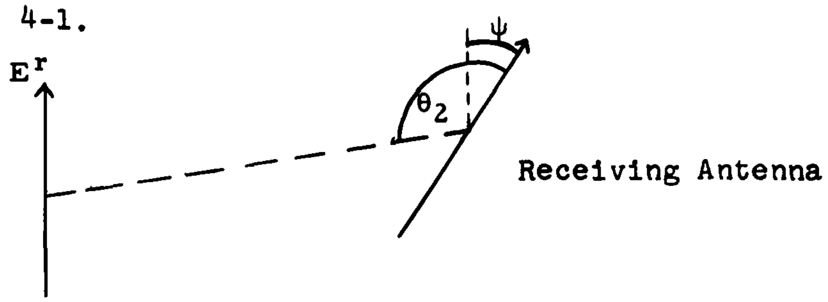

Fig. 4-1. Dipole antenna in an arbitrary plane wave field

The King-Middleton integral equation for the receiving antenna in an arbitrarily orientated plane wave field is:

$$\int_{-h}^{h} I_Z(Z^1)k_d(Z,Z^1) = -j\frac{4\pi}{\zeta}\Big[\mho(\cos kZ-\cos kh)$$
$$+\mho^1(\cos qh \cos kZ-\cos qZ \cos kh) \qquad (4\text{-}2)$$
$$+\tfrac{1}{2}V \sin k(h-|Z|)\Big]$$

where

$$K_d(Z,Z^1) = \frac{e^{-jkR}}{R} - \frac{e^{-jkR_h}}{R_h} = K(Z,Z^1) - K(h,Z^1)$$

$k = \beta - j\alpha$, β = phase constant, α = attenuation constant (4-3)

$R = \sqrt{(Z-Z^1)^2 + a^2}$, a = radius of antenna

$R_h = \sqrt{(h-Z^1)^2 + a^2}$

$$\mho = -j\frac{\zeta}{4\pi}\int_{-h}^{h} I_Z(Z^1)K(h,Z^1)dZ^1,\ \mho^1 = -E^Y \cos\Psi/k \sin\theta_2$$

$V = I(0)Z_2$ for the receiving case, $I(0)$ = base current (4-4)

Z_L = Load impedance, $q = k \cos\theta$.

The right hand side of the integral equation (4-2) contains three terms: the first is proportional to $[\cos kz - \cos kh]$, where the current is on a dipole in a parallel plane wave field. The coefficient U is constant along the length of the antenna, where the plane wave is parallel to the element. The second term has a coefficient proportional to the incident electric field. This term produces a current proportional to $[\cos qh \cos kz - \cos qz \cos kh]$. The complexity of the second current compared to the first is due to the tilt of the incident electric field. When the incident electric field is parallel to the antenna, this

term reduces to (cos kz-cos kh). The last term on the right hand side of eq. (4-2) is proportional to the equivalent driving voltage and produces a sinusoidal current as in an open wire line or an isolated transmitting antenna.

The solution for the electrically short antenna follows for the expansion of the integral with β_h as a small parameter and the new quasi-zeroth order solution for the King-Middleton Integral equation.*

The original integral equation (4-2) may be rewritten for the electrically short receiving antenna as follows:

$$\int_{-h}^{h} I_Z(Z^1)\left[\frac{1}{R} - \frac{1}{R_h} \quad \frac{h^2}{2}(R-R_h) + j\,\frac{h^3}{b}\,(R^2-R_h{}^2)\right]dZ^1 =$$

$$= -\frac{j2\pi k^2 h}{\omega\,\mu(1-\frac{k^2h^2}{2})}\left[\mathcal{U}\,Ckh(1-\frac{Z^2}{h^2}) + \mathcal{U}'C'kh(1-Z^2/h^2) + V\,S(1-|Z|/h)\right]. \qquad (4-5)$$

where

$$\cos kz - \cos kh \cong \tfrac{1}{2}\,Ck^2\,(1-Z^2/h^2) = F_{oZ}$$

$$\cos qh \cos kz - \cos qz \cos kh \cong \tfrac{1}{2}\,C\,k^2h^2(1-Z^2/h^2) = F_{oZ}, \qquad (4-6)$$

---

*A similar solution for the transmitting antenna may be found in "The electrically short antenna as a probe for measuring electron densities and collision frequencies in an ionized medium," R. King and C. Harrison, J. of N.B.S., D, 65, No. 4, pp. 371-384, July-August 1961.

$$C = 1-(k^2h^2/12)$$
$$C^1 = (1 - \frac{k^2h^2\sin^2\theta}{12}) \frac{\sin^2\theta}{2} \qquad (4\text{-}7)$$
$$S = 1 - (k^2h^2/6)$$

The integral equation (4-) with $C^1 = 0$ has been solved by King, *et al.* [10] by using suitable trigonometric approximations for the vector potentials produced by the triangular $(1- |z| /h)$ and parabolic $(1-z^2/h^2)$ current distributions. Expressions for the driving point impedance (Z_0) and admittance (Y_0) of the transmitting antenna are given by King. The additional current introduced by the impinging plane wave field is given by:

$$I_Z^r(Z) = j\frac{2\pi k}{\Psi_{d1}\omega\mu(1-\frac{k^2h^2}{2})} \frac{V^1 F_{oZ}^1}{\Psi_{d1}(1-\frac{\quad}{2}) - \Psi_u(h)} \qquad (4\text{-}8)$$

where

$$\Psi_{d1} \cong 2\ \ell n\ (h/a) - 2$$
$$\Psi_{du} \cong 2\ \ell n\ (2h/a) - \beta + 5/12\ k^2h^2 - j\ 2/a\ k^3h^3$$
$$\Psi_u(h) \cong R^2h^2 - j\ 2/3\ R^3h^3 \qquad (4\text{-}9)$$
$$F_{oZ}^1 \cong C^1kh\ (1-Z^2/h^2)$$
$$k = \beta - j\ \alpha\ .$$

The receiving part of the antenna current $I_z^r(z)$ reduces to the following approximate form with eq. (4-9) in eqn. (4-8):

$$I_z^r(z) = \frac{2\pi\beta^2 h^2 E^r \cos\Psi \sin\theta}{\zeta_e(2\ln\frac{2h}{a} - 3)} \left(\frac{\alpha}{\beta}\right)^3 - 3\left(\frac{\alpha}{\beta}\right) - j\left(1 - 3\,\frac{\alpha^2}{\beta^2}\right)\left(1 - \frac{z^2}{h^2}\right) \quad \text{(4-10)}$$

where

$$\zeta_e = \zeta_0/\sqrt{\epsilon_r} = 120\pi/\sqrt{\epsilon_r},$$

$$\frac{2\alpha}{\beta} = \frac{\sigma}{\omega\epsilon_0\epsilon_r} \quad \text{(4-11)}$$

The Thévenin equivalent circuit may now be constructed for the receiving antenna. The equivalent circuit is shown in Fig. (4-2).

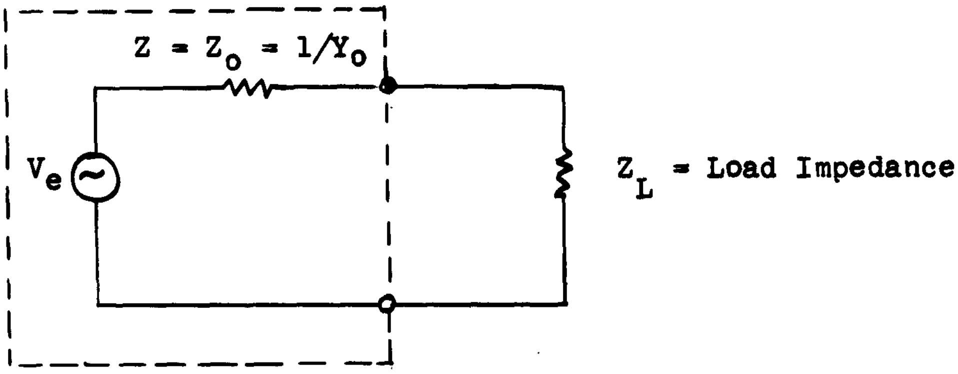

Fig. (4-2). Equivalent Circuit for Receiving Antenna.

The equivalent impedance Z is equal to the impedance seen at the antenna terminals with all generators short circuited (i.e., $V_e = 0$ or $E^r = 0$), thus:

$$Z = Z_0,\ Y_0 = 1/Z_0 = G_0(h) + j\,B_0(k)$$

$$Y_0 = \frac{2\pi}{\zeta_e \Psi_{dl}} \left\{ \frac{2\alpha}{\beta}\left[\beta h + \frac{2}{3}\beta^3 h^3 F\right] + \frac{\beta^4 h^4}{3(\Omega - 3)} \right. \quad \text{(4-12)}$$

$$\left. + j\left[\beta h + \frac{1}{3}\beta^3 h^3 F - \frac{5}{3}\,\frac{\alpha}{\beta}\,\frac{\beta^4 h^4}{(\Omega - 3)}\right] \right.$$

where

$$\left(\frac{2a}{\beta}\right)^2 \ll 1$$

$$F = 1 + \left[\frac{1.08}{\Omega - 3}\right].$$

The equivalent voltage of the receiving antenna V_e is the open circuit voltage at the antenna terminals (i.e., $Z_L = \infty$), thus:

$$V_e(Z_L = \infty) = I_Z{}^r(Z_L = 0)Z_o \qquad (4\text{-}13)$$

$$V_e = \frac{2\pi\beta^2 h^2 \cos\Psi \, Z_0 \sin\theta}{\zeta_e \left(2 \ln \frac{2h}{a} - 3\right)} \quad \left(\frac{a}{\beta}\right)^3 - 3\left(\frac{a}{\beta}\right) - j\left(1 - \frac{\;}{\beta^2}\right) \qquad (4\text{-}14)$$

5. Theoretical determination of the electron density and collision frequency of the plasma from measurements on the receiving antenna

The general procedure is to measure the equivalent impedance of the receiving antenna Z_o, due to the incoming signal. The incoming wave is assumed to propagate parallel to the magnetic field under Z-mode conditions. From the measurement of Z_o the dielectric constant ϵ and conductivity are determined. The values of ϵ and σ are then related to the electron density N and collision frequency ν through the Appleton-Hartree equations. The Appleton-Hartree equation represents the propagation factor for a plane (actually circularly polarized) wave operating in the Z-mode. The

dielectric constant and conductivity for a left handed circularly polarized wave propagating parallel to the magnetic field are given by [11]:

$$\epsilon = \epsilon_o \epsilon_i = \epsilon_o \left[1 - \frac{Ne^2}{\epsilon_o m\omega} \cdot \frac{(\omega - \omega_b)}{\nu^2 + (\omega - \omega_b)^2} \right] \tag{5-1}$$

$$\sigma = \frac{Ne^2 \nu}{m\nu^2 + (\omega - \omega_b)^2} \tag{5-2}$$

where

$$\omega_b = \left| \frac{\epsilon}{m} B_o \right|, \quad B_o = \text{static mag. field,} \tag{5-3}$$

When the above expressions are simultaneously solved for N and ν the result is given by:

$$N = \frac{m\omega \epsilon_o (1 - \epsilon_r)}{e^2} \cdot \frac{\nu^2 + (\omega - \omega_b)^2}{(\omega - \omega_b)} \tag{5-4}$$

and

$$\nu = \frac{\sigma(\omega - \omega_b)}{\omega(\epsilon_o - \epsilon)} \quad . \tag{5-5}$$

The values of $\epsilon_r = \epsilon / \epsilon_o$ and σ are determined by the experimental measurement of Y_o (= $1/Z_o$) from the formulas given by King, [12] or:

$$\epsilon_r = \frac{B(k)}{B(\beta_o)} \left[1 + \frac{1}{3} F \beta^2 h^2 \left(1 - \frac{B(k)}{B(\beta_o)} \right) \right] \tag{5-6}$$

and

$$\frac{2\alpha}{\beta} = \frac{\sigma}{\omega \epsilon_o \epsilon_r} = \frac{G(k) - \epsilon_r^{5/2} G(\beta_o)}{B(k)(1 - 1/3 F \beta^2 h^2)} \tag{5-7}$$

where

$$Y(\beta_o) = G(\beta_o) + jB(\beta_o)$$
$$Y(k) = G(k) + jB(k) \;. \qquad (5\text{-}8)$$

The experimental measurement of Z_o is made by successively attaching two lengths of transmission line to the terminals of the receiving antenna. The voltage along the transmission line is first sampled at $x = x_1$ with a line length ℓ_1 and then at $x = x_2$ with a line length ℓ_2. Using the elementary transmission line theory it is possible to relate these two measurements to the impedance Z_o.

The transmission line equations for a lossless line yield the following expressions for the voltages at the two sampled points: (N.B. Z_o = characteristic impedance of transmission line):

$$V_{x_1}(\ell_1) = \frac{Z_o V_e \left[Z_L \cos\beta(\ell_1 - x_1) + j\zeta_o \sin\beta(\ell_1 - x_1) \right]}{\zeta_o (Z_o + Z_L)\cos\beta\ell_1 + j(Z_o Z_L + \zeta_o^2)\sin\beta\ell_1} \qquad (5\text{-}9)$$

and

$$V_{x_2}(\ell_2) = \frac{Z_o V_e \left[Z_L \cos\beta(\ell_2 - x_2) + j\zeta_o \sin\beta(\ell_2 - x_2) \right]}{\zeta_o (Z_o + Z_L)\cos\beta\ell_2 + j(Z_o Z_L + \zeta_o^2)\sin\beta\ell_2} \;. \qquad (5\text{-}10)$$

It follows from eq. (5-9) and eq. (5-10) that the ratio of the two sample voltages is:

$$\frac{V_{x1}}{V_{x2}} = \frac{Z_L \cos\beta(\ell_1 - x_1) + j\zeta_o \sin\beta(\ell_1 - x_1)}{Z_L \cos\beta(\ell_2 - x_2) + j\zeta_o \sin\beta(\ell_2 - x_2)} \cdot \frac{\zeta_o (Z_o + Z_L)\cos\beta\ell_2 + j(Z_o Z_L + \zeta_o^2)\sin\beta\ell_2}{\zeta_o (Z_o + Z_L)\cos\beta\ell_1 + j(Z_o Z_L + \zeta_o^2)\sin\beta\ell_1} \qquad (5\text{-}11)$$

or

$$V = A_1 \cdot \frac{Z_o A_2 + A_3}{Z_o A_4 + A_5} \tag{5-12}$$

where

$$A_1 = \frac{Z_L \cos\beta(\ell_1 - x_1) + j\mathfrak{z}_o \sin\beta(\ell_1 - x_1)}{Z_L \cos\beta(\ell_2 - x_2) + j\mathfrak{z}_o \sin\beta(\ell_2 - x_2)}$$

$$\begin{aligned} A_2 &= \mathfrak{z}_o \cos\beta\ell_2 + j\, Z_L \sin\beta\ell_2 \\ A_4 &= \mathfrak{z}_o \cos\beta\ell_4 + j\, Z_L \sin\beta\ell_4 \\ A_3 &= \mathfrak{z}_o Z_L \cos\beta\ell_2 + j\mathfrak{z}_o^2 \sin\beta\ell_2 \\ A_5 &= \mathfrak{z}_o Z_2 \cos\beta\ell_1 + j\mathfrak{z}_o^2 \sin\beta\ell_1 \\ V &= V_{x1}/V_{x2} \end{aligned} \tag{5-13}$$

The above formulas may be simplified if the probes are arranged such that:

$$\ell_1 - x_1 = \ell_2 - x_2 \Rightarrow A_1 = 1, \tag{5-14}$$

and therefore

$$V = \frac{Z_o A_2 + A_3}{Z_o A_4 + A_5} \; ; \tag{5-15}$$

or solving for Z_o,

$$Z_o = \frac{A_3 - VA_5}{VA_4 - A_2} \; . \tag{5-16}$$

The effective area is computed by finding the power dissipation in the load impedance Z_L due to the incoming wave of magnitude E^r, or:

$$P_L = \frac{Re}{2} V_e^2 \frac{Z_L}{(Z_L + Z_o)^2} = \tfrac{1}{2} AeS \tag{5-17}$$

where

A_e = effective area

S = power density in wm^2 .

Thus from eq. (5-14) in eq. (5-17) the power in the load impedance is:

$$P_L = \mathrm{Re}\ \frac{2\pi\beta^2 h^2 \cos\psi Z_o \sin\theta}{\zeta_e (2\ln\frac{2h}{a} -3)} \left|\left[(\frac{\alpha}{\beta})^3 - 3(\frac{\alpha}{\beta}) - j(1 - 3\frac{\alpha^2}{\beta^2})\right]\right|^2 \frac{|E_r|^2}{2} \frac{Z_L}{(Z_L + Z_o)^2} \tag{5-18}$$

The power density S in watts per square meter is given by:

$$S = \tfrac{1}{2} E \times H^* = \frac{\zeta}{2} E^2 = \frac{(1 - j\frac{\alpha}{\beta})E^2}{2\zeta_e} \tag{5-19}$$

By inspection, the effective area is:

$$A_e = \mathrm{Re}\ \frac{2\pi\beta^2 h^2 \cos\psi \sin\theta Z_o}{(2\ln\frac{2h}{a} -3)} \left|\left[(\frac{\alpha}{\beta})^2 - 3(\frac{\alpha}{\beta}) - j(1 - 3\frac{\alpha^2}{\beta^2})\right]\right|^2 \frac{Z_L}{(Z_L + Z_o)^2}$$

For a matched load $Z_L = Z_o^*$, or $Z + Z_o = 2R_o$, and therefore:

$$A_{e(opt.)} = \frac{2\pi\beta^2 h^2 \cos\psi \sin\theta Z_o^*}{(2\ln\frac{2h}{a} -3)\ 2R_o} \left|\left[(\frac{\alpha}{\beta})^2 - 3(\frac{\alpha}{\beta}) - j\ (1 - \frac{3\alpha^2}{\beta^2})\right]\right|^2$$

for $(\frac{\alpha}{\beta}) \ll 1$

$$A_{e(opt.)} = \left|\frac{2\pi\beta^2 h^2 \cos\psi \sin\theta}{(2\ln\frac{2h}{a} -3)} \cdot \frac{Z_o^*}{2R_o}\right|^2 .$$

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